

Homotopy approach to quantum gravity

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Abstract

I construct a finite-dimensional quantum theory from general relativity by a homotopy method. Its quantum history is made up of at least two levels of fermionic elements. Its unitary group has the diffeomorphism group as singular limit. Its gravitational metrical form is the algebraic square. Its spinors are multivectors.

1 Strategy

I report here on progress in the search for a physical theory that covers both general relativity and quantum theory. Segal 1951 made a key contribution when he pointed out that both theories made a basic classical Lie algebra simple or at least simpler by a homotopy that introduced a small non-commutativity, with homotopy parameters having as their final values the physical constants \hbar and c . He proposed to carry this to its logical limit, a homotopy to a simple Lie algebra, requiring additional parameters. Classical physics flattens this simple Lie algebra. The strategy of full quantization is to “flex” it back to its natural simple form. Canonical quantization in a nutshell is a homotopy that “replaces Poisson Brackets by commutators”. Full quantization similarly compressed is a homotopy that “replaces commutator Lie algebras by simple Lie algebras”. Einstein and Heisenberg effected three homotopies, with parameters c , G , and \hbar . Full quantization extends all three homotopies to one that leads to a simple Lie algebra.

One may explain the drift toward a simple Lie algebra, and thus both relativization and quantization, in Darwinian terms. Within the class of Lie

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groups, the simple Lie groups are stable against small changes in the structure tensor, and the Lie groups of classical mechanics and canonical quantum mechanics are compound (not semi-simple) and are not stable. In the population of competing theories, stable ones probably survive longer and have more offspring than unstable ones. Flexing stabilizes, and so promotes survival.

Each epoch defines its own stability construct. For example, Segal stabilized Lie algebras against variations in the Lie product but not in the Jacobi identity or co-commutativity. I follow the same strategy because it has not yet been fully executed for any physical theory.

When we smash a regular algebra flat, it becomes singular and unstable. One can make it finite by chopping it into bits of our choosing— discrete elements — and discarding all but a finite number, but this removes it even further from nature. Simple algebras have finite-dimensional representations giving all dynamical variables discrete bounded spectra. Such a quantum theory decides its own finite quantum elements. In some cases, one flexing can quantize the theory, relativize it, stabilize it, and regularize it. It does not preserve the symmetry group of the flattened structure identically like renormalization, nor break it completely like lattice regularization, but flexes it, and so slightly that it continues to fit past experiments.

1.1 Flexing

Let Λ be the manifold of Lie algebra structure tensors on a fixed vector space V . Let $C^n = \{p^k \mid 0 \leq p^k \leq P^k\} \subset \mathbb{R}^n$ be a closed n -cell at the origin $\mathbf{0}$ with edges P^k .

Definition 1 *A flex $L_0 \leftarrow L_1$ is an n -parameter homotopy $h : C^n \rightarrow \Lambda$ with unstable $h(\mathbf{0}) = L_0$, $h(X^k) = L_X$, and simple $h(p)$ when all $p^k > 0$.*

This is an inverse process. The direct process is the contraction, flattening, or singular limit $\lim_{\mathbf{0}} S = L$ (or $L \rightarrow S$). Flexing a Lie algebra of observables is a form of quantization, which may be called *full quantization* to distinguish it from canonical quantization, which is always partial.

Note that people paradoxically call the unbounded structure a “contraction” of the bounded, and call the truer structure a “deformation” of the flattened one.

1.2 Background

I came to this problem as follows. Since 1997 I have taken seriously a universal relativity: that every construct of physics is relative to the experimenter and changed by its determination. But every physical theory begins with postulates

containing allegedly absolute constructs. Therefore every physical theory has limited validity, by the nature of the enterprise. The goal of physics is therefore not a final theory but the next step in an on-going Darwinian process of theory-selection. The universe will probably always surprise us.

The search for some theoretical framework for this process sent me back to the classic work of Inonu and Wigner 1952 on group contraction, which now sent me to the work of Segal 1951 on the inverse process to group contraction, here called flexing. In his sole illustration, Segal flexed the Heisenberg commutation relations for one coordinate variable q and momentum p to orthogonal group relations

$$\lim_{\alpha, \beta \rightarrow 0, r \rightarrow 1} (qp - pq = i, \quad iq - qi = 0, \quad pi - ip = 0) = (qp - pq = r, \quad rq - qr = \alpha p, \quad pr - rp = \beta q), \quad (1)$$

adjoining and thawing a variable r frozen in the singular limit.

Segal's proposal influenced the study of contraction and the Galilean limit by Inönü and Wigner 1952, and the Gerstenhaber 1964 cohomological theory of Lie algebra stability. These in turn had many effects, many of them first brought out for me at this meeting (Oberwohlfach 2006).

Vilela Mendes 1994 first applied the stabilization strategy to a relativistic physical theory. He stabilized the Poincaré algebra and the Heisenberg algebra of Minkowski space, using a fundamental length that could well be the one that Heisenberg prophesied a half-century earlier. His quantum space is a matrix geometry in the sense of Dubois-Violette *et al* 1989 except for the theory of connections and gravity. It is more matricial than the Matrix Model of Banks 1977 in that its time variable too is a matrix, and it goes beyond the deformation quantization of Flato 1982 in that its end Lie algebra is simple.

Vilela space combines and unifies not only the homotopies of Einstein and Heisenberg, but also those of de Sitter, Snyder, and others who navigated without the stability compass and so never reached simplicity. It has the simple group $SO(6; \sigma)$ of a quadratic space with somewhat unspecified signature σ .

To apply the stabilization strategy to statistics I define:

Definition 2 *A paleo-bosonic statistics is one defined by a simple Lie algebra having the bosonic Lie algebra as singular limit.*

Then the Vilela quantum space is an aggregate of paleo-bosonic sub-events that have a real six-dimensional ket space of non-Euclidean signature (§2.9). Its event coordinate operators $\hat{x}, \hat{p}, \hat{i}$ represent elements of $dSO(6; \sigma)$. The dimension of the representation is unspecified but is presumably allowed to be large, since Vilela 1994 uses an infinite-dimensional representation for a singular limit.

The dynamics of linear systems like the harmonic oscillator is expressible by a Lie algebra and is therefore flexible. Shiri-Garakani and Finkelstein 2005 constructed a stable dynamics for such a system by flexing the usual unstable one. They find that different time-eigenvalues define subspaces of different dimensionalities, so there is no one-parameter unitary group of time translations, except in the singular limit of classical space-time. Near the beginning and end of a system time t , when $|t| \sim \pm lX$, the multiplicities of the eigenvalues of $|t|$ vary linearly in $|t| - \max ||t|$ and unitarity is a bad approximation. This makes room for a quantum version of the black hole. In the middle times, $|t| \ll \max ||t|$, unitarity is a reasonable approximation.

Baugh 2004 flexed the Poincaré group to an A group, independently of the work of Vilela Mendes with a D group. The quantum event of Baugh space is an aggregate of paleo-bosonic sub-events with a six-dimensional complex ket space. For possible future needs I generalize Baugh space from dimension 6 to an extended event space of any dimensionality ν and signature σ . The event coordinate operators $\hat{x}, \hat{p}, \hat{t}$ then represent elements of $dSU(\nu; \sigma)$.

1.3 Outline

In §2 I extend full quantization to the history Lie algebra, as required for general relativity, formulating a strategy of general quantization. In §3 I generally quantize the Einstein space, algebra, and group. In §4 I generally quantize the Einstein kinematics and discuss the spin-statistics correlation. In §5 I describe a general quantum gravity.

I stop in the middle of the work. It remains to reconstruct classical general relativity as singular limit of this quantum theory, verifying the heuristic arguments used to construct the theory, and to work out the experimental consequences that differ from the classical theory. The main indication that the theory worked would be getting some of the particle spectrum and forces right. The immediate problem is to get any particles and forces at all, for they live one or two levels above the level of events.

2 General quantization

2.1 Foreground

Heisenberg, emulating Einstein, set out to work solely with observables, and ultimately his quantum theory encodes operations of observation in single-time operators $Q(0)$. But Heisenberg's dynamical equations $dQ(t)/dt = [H(t), Q(t)]$ do not relate these alleged observables, they relate observable-valued functions

of time $Q(t)$, histories. A history of an observable is not an observable in the Heisenberg sense. When he works with admittedly non-observable entities Heisenberg does not yet strictly conform to his own operational principle. In a regular theory we can come closer to operationality with the following strategy:

Attach observables to histories, not instants.

To formulate quantum dynamics in classical space-time in a covariant way, Dirac, Schwinger, and Feynman already replaced probability amplitudes for the instantaneous system by probability amplitudes for system histories. I do the same for quantum dynamics in a quantum space of events. Continuum theories of history have been singular in the extreme, with operations like “integrating over all histories” that were programmatic rather than defined mathematical processes. Flexing, however, makes the quantum history more docile than a quantum crystal. In the fully quantum theory, sums over all histories are merely traces of finite-dimensional matrices, while for crystals today the analogous matrices are infinite-dimensional and almost all the traces diverge.

To form a general quantum theory I construct a classical history Lie algebra (§2.2), flex it (§2.3), and represent it in a Fermi algebra (§2.10). These steps have singular correspondents in canonical quantization, which is an initial stage of general quantization that does not yet require us to append frozen variables. Instead of a Fermi algebra representation, the Heisenberg Lie algebra uses the infinite-dimensional representation R_{\hbar} fixed by Planck’s constant. Its unitary flex has instead an infinite number of finite-dimensional non-faithful irreducible representations R_J fixed by weights. The Clifford and Fermi algebras that arise in general quantization define their own unique faithful irreducible representations up to isomorphism.

2.2 Unified history

To start with the easiest fully quantum relativistic dynamics, Finkelstein 2005 flexed a real scalar meson field. The point of this example was to quantize the history Lie algebra as a unit, not the field variables and the space-time variables separately. I recapitulate some necessary concepts:

Definition 3 *A q/c system is one with quantum variables and classical commuting time and possibly space coordinates, as in canonical quantum theory; c/c and q/q systems are similarly defined.*

A c/c history space is the space of all possible state functions $(q(t), p(t))$.

A c/c history algebra is the algebra of all scalar functions of the history.

A c/c history Lie algebra is the trivial commutator algebra of the history algebra, which is commutative.

*In the **q/c case**, replace real-valued functions $q(t), p(t)$ by operator-valued functions in the above definitions.*

*In the **q/q case**, the history space is a quantum space defined by its coordinate algebra, a flex of a q/c or c/c one. A history ket defines a fully quantum dynamics by assigning a probability amplitude to any other history ket.*

To flex classical general relativity I retrace Einstein's path to gravity at the full quantum level of resolution, one or two levels beneath classical space-time, starting from the equivalence principle and flexing as I go.

Invariance under the Einstein group, the principle of general covariance, incorporates the equivalence principle and led Einstein to describe gravity with the chronometric quadratic form $g_{\mu\nu}$. The Einstein group is defined by singular relations like $[\partial_\mu, x^\nu] = \delta_\mu^\nu$, so it is an unstable compound group, ripe for flexing.

Unlike canonical quantization, flexing is not intended to preserve general covariance and the classical Einstein group exactly. General covariance is classical and unstable. Flexing replaces it with a stable quantum correspondent, general quantum covariance.

2.3 Unified algebra

Quantum theory has one product while general relativity has many. Part of the solution is inspired by conversations with Aage Petersen 1970:

Quantization merges products.

This already happens in canonical quantization, which merged (1) the commutative algebraic product of functions on phase space and (2) the non-associative Poisson bracket product of the same functions. Heisenberg recovered both from the non-commutative product of quantum mechanics as leading terms in an \hbar power series.

General quantization merges (1) the commutative inner product $v \cdot w$ and (2) the anti-commutative Lie product $[v, w]_{\text{Lie}}$, two products of space-time vector fields. These derive from two associative products on the vector fields: the differential-operator product and the Clifford product. I merge them into one Fermi algebra of general quantum gravity, which is also the operator algebra of the quantum system, and a Clifford algebra. The history ket space of the system is then a multivector space for this Fermi algebra, and a spinor space for this Clifford algebra.

A further plurality of products in the theory arises from the plurality of levels.

2.4 No-field theory

Hilbert varied gravitational field variables $g_{\mu\nu}(x)$ without varying coordinates $x = (x^\kappa)$. In the resulting Poisson Bracket Lie algebra, $g_{\mu\nu}$ commutes with x^κ . There are no such coordinates in real life. The lattice of rods and clocks imagined by Einstein provides such coordinates at low resolution but would obliterate the system at high resolution. Our actual physical coordinates x^μ are all based on weak signals, usually electromagnetic, that carry us information about the intervening gravitational field as well as the remote event, as in the first astronomical observations of the solar deflection of star images. Such coordinates are more relative than general relativity, relative to the field as well as to the frame of reference. Coordinates in the small that commute with each other and the gravitational field are unnatural in the canonical theory too, as Bergmann and Komar 1972 and Bergmann 1979 point out. Once coordinates fail to commute we can dispense with quantum fields, which re-emerge in the singular limit of classical space-time. One way to unify fields is to eliminate them all together.

As a classical prototype, the gravitational field emerges from a no-field theory that imbeds space-time in a higher-dimensional flat space. In the quantum correspondent, we imbed a quantum space of actual events within a quantum space of possible events.

Quantum logic originally forced me to avoid the usual field concept. Quantum logic has an invariant construct of subset but no general construct of *functional* relation between given quantum variables, as discussed in Finkelstein 1969. In general quantum gravity the main variable, the history, is a quantum set of actual events, a variable subset of a fixed quantum space of possible events. In fully quantum gravitational dynamics, gravity is already present in the quantized event coordinates.

The quadratic form of general quantum gravity is defined by the Fermi-algebra product, which is also a Clifford-algebra product. If $v = \lim_{\mathbf{0}} \hat{v}$ is a classical vector field and the singular limit of an operator \hat{v} in the Fermi algebra, then the value of the gravitational operator-valued quadratic form is

$$\begin{aligned} g(v) &= g_{\mu\nu}(x)v^\mu(x)v^\nu(x) \\ &= \lim_{\mathbf{0}} (\hat{v})^2. \end{aligned} \tag{2}$$

Clifford algebra guides our general quantization much as the Poisson Bracket Lie algebra guides canonical quantization.

2.5 Unified statistics

The quantum event of Vilela space has a paleo-bosonic coordinate algebra of x and ∂_x , as though the event itself is a paleo-bosonic aggregate. And the space is cold, so most of its constituents could occupy one ket. If the Vilela Mendes quantization is correct, it is odd that history does not consist of one ground event that occurs very many times and many rare events. Instead history behaves like a fermionic aggregate; even a crystal (as Newton noted) with its transverse waves. Crystals are stabilized against collapse by the fermionic statistics of electrons; I stabilize history by a similar strategy:

Analyze paleo-bosonic events into fermionic ones.

(§2.13, §2.10). This also permits a formulation of the spin-statistics correlation that makes it natural to extend it to other levels. The model studied here is fermionic to its bottom, several levels of quantification down.

Standard quantum field theory works with at least the following successive levels of aggregation, listed from the top down:

1. the many-quantum or field operator history $\check{\psi}(x)$,
2. the single-quantum ket $\psi(x)$,
3. the coordinate $x = \int dx$,
4. the differential dx .

Quite different bridges connect these levels. We pass from 4 to 3 by integration, from 3 to 2 by quantization, and from 2 to 1 by quantification. Each of these levels has its own algebraic structure, 4 and 3 being classical and 2 and 1 quantum. This arrangement seems unphysical, since surely the microworld is quantum.

For the purely classical field theory of the 19th century one mode of aggregation, set formation, would have sufficed to express all aggregates and bridge between them. Canonical quantization must us classical modes of aggregation on some levels and quantum on others. General quantization reopens the possibility of a uniform aggregation process or statistics, now quantum instead of classical.

Definition 4 (M and μ) *If S is a system (classical or quantum) then MS designates an aggregate whose generic element is S . M is for **Many**-, **Meta**-, or **Menge**. If A is an aggregate then in general μA designates the generic element or quantum of A ; μ is for **mini**-, **mero**-, or **micro**-. μ^n and M^n designate n -fold iterates of μ and M .*

*VS designates the **ket space** of S , with norm $\|\psi\| = \psi^\dagger \psi$ defined by a hermitian form \dagger .*

A S designates the **coordinate** \dagger **algebra** of system S , the endomorphism algebra of VS .

The multivalued operation Min accepts an algebra and returns a **minimal left ideal** of the algebra.

The prefix M can designate any quantification process depending on context or labeling. In a given context μ and M are inverses of each other (as in the metric system). I designate the history that we analyze by H , and its constituent events by $E = \mu H$.

Assumption 1 *There is one **basic quantifier** M and it is fermionic.*

This is a typical assumption of the uniformity of nature, based on solid ignorance. The quantum event is still out of reach, the chronon more so. We have a chance of describing it correctly only if it repeats what we find on the higher levels. At least this assumption does not require us to express bosons like photons in terms of fermions like neutrinos, as de Broglie proposed. I propose to analyze all the gauge fields as I analyze gravity, into fermionic quantum events, not particles.

Segal's three variables p, q, r generate the algebra $d\text{SU}(2) = d\text{SO}(3)$, in both the A and B series. In higher dimensions, however, one must choose between the A series of groups $\text{SU}(n)$ on complex vector spaces and the B and D series of orthogonal groups SO on real vector spaces. This choice must correlate with the choice of statistics. There are some tentative indications that the relativity group is in the A series and that there is a basic Fermi statistics:

1. The i -invariance of ordinary complex quantum theory breaks down in the real B and D series and survives in the A series.
2. The singular limit that recovers classical mechanics from quantum mechanics automatically converts the complex theory into a real one.
3. Four obvious candidates for a basic quantifier are Cliff, Min Cliff, Fermi and Min Fermi. Iterating Cliff or Fermi violates the spin-statistics correlation.
4. Min Cliff has a fixed point at dimensions 2 and 4, blocking analysis into binary elements.
5. Min Fermi = Grass = \bigvee = M is singled out because it respects the spin-statistics correlation and permits analysis into binary elements.
6. The internal groups of the Standard Model are all unitary but not all orthogonal groups, thanks to $\text{SU}(3)$.

7. The paradigm of universal quantification theories, classical finite set theory, is an iterated fermionic statistics over the binary field of scalars 2.
8. Fermi quantification has a stable group.
9. Fermi quantification accounts for both Fermi and Bose statistics at once (§2.11), and also for spin (§2.10).
10. The Spin Lie algebra of Minkowski space-time is the Lie algebra of an orthogonal group, not a unitary group; but it is a singular limit of a unitary group, in the same limit that produces classical space-time.
11. Classical gravity is described by a Clifford ring; but this is a singular limit of a Fermi algebra. (§4.1, §5).

The last two indicators originally pointed to the D series but I have re-aligned them by *ad hoc* assumptions to secure consistency.

The Kaluza-Klein strategy imbues space-time with extra internal compact dimensions. This created the compactification problem: What compactifies these dimensions? Einstein and Mayer 1931 evaded this problem by adding extra components to the tangent vectors of the c space-time manifold but no dimensions to the base manifold itself. Connes 1994 makes a similar theory in a mixed c - q event space, attaching quantum spinlike variables to a classical manifold. Assembling history from fermionic quantum events permits me to emulate them at a fully quantum level. One must posit that these quantum events bind along some dimensions of their ket space to form the macroscopic space-time dimensions, leaving all other dimensions small, on the chronon scale, like a soap bubble that is only one molecule thick in one direction but macroscopic in the other three space-time directions.

The compactification problem is replaced by the extension problem: What makes some dimensions extend to macroscopic sizes? As with soap bubbles, this is a matter of the structure of the molecular elements. I do not reach this problem here.

2.6 Regularization

Almost all quadratic forms are regular, almost all matrices have inverses, almost all determinants are non-zero. The divergent cases are exceptional, rare. Therefore any singular theory is not based entirely on experiment, which is always generic, but also on belief in some occurrence of zero probability. Flexing eliminates such singularities as the Wronskian singularity of gauge theories and the singularities of propagators. Instead of infinite renormalizations flexing introduces quantum constants that are up front and finite. Vilela space has three new homotopy parameters and quantum constants: a space quantum X ,

a momentum quantum P , and a large quantum number N , in addition to the usual quantum of action and angular momentum \hbar .

The scalar meson in Minkowski space-time and general relativity both have Lie algebras which are infinite dimensional because their elements depend on arbitrary functions, for example functions of time. General quantization replaces these Lie algebras by ones of high but finite dimensionality.

2.7 System structure

To structure the history H we must analyze aggregates into elements more than once. Present quantum field theory uses several traditionally distinct aggregation processes:

1. *Summation* makes a whole that is “the sum of its parts,” at least for some properties such as charge.
2. *Integration* generates quantities from their differentials.
3. *Quantification* creates an aggregate of elements. The logician William Hamilton introduced “quantification” to transform yes-or-no questions about an individual into how-many questions about an aggregate.
4. *Exponentiation* creates F^X as an aggregate of F -systems, one at each point of the set X .
5. *Group generation* creates finite transformations from infinitesimal ones. This constructs groups from their infinitesimal Lie algebras.
6. *Quantization* can be regarded as atomization followed by quantification (§2.8).

In general quantum relativity all these are provisionally replaced by one aggregation process.

2.8 Quantization and quantification

Quantification in the quantum domain, usually Bose or Fermi, formally resembles quantization so closely that at first it was called “second quantization”, although quantification converts c systems into c , and q systems into q , while quantization converts c systems into q . Nevertheless the two are related.

Assertion 1 *Quantization is quantum atomization followed by quantification.*

Argument (heuristic). Quantization begins by forming a Lie algebra. The vector space supporting the Lie algebra can be interpreted as the input/output

vector space of a constituent or “atom” of the system. The relations of the Lie algebra define a quantification for this atom. \square

For example, to canonically quantize the linear harmonic oscillator, with coordinate x and momentum p , one can:

(1) Posit a boson “atom” B whose sole attribute is existence, with one-component complex normalized kets $|B\rangle$ and bras $\langle B|$; and then

(2) Quantify B , forming a bosonic aggregate MB with two dimensionless Lie-algebra generators, a B -creator $b^\dagger = \langle \beta|B\rangle$ and a B -annihilator $b = \langle B|\beta\rangle$, subject to

$$[b^\dagger, b] = c \quad (3)$$

where c is a third basis element that is central. Here $\langle \beta|$ is an operator-valued form converting kets into bosonic creation operators. Finally one chooses a representation. In this example the representation is fixed by the value it assigns to the central invariant $c = i\hbar$. The Heisenberg relations are satisfied by

$$x = X \frac{(b + b^\dagger)}{2}, \quad p = P \frac{(b - b^\dagger)}{2i} \quad (4)$$

X, P are dimensional constants required for homogeneity of units, with $XP = \hbar$. The classical limit is $\hbar \rightarrow 0$.

Quantum electrodynamics too has been quantized by (1) atomization and (2) quantification. Akhiezer and Berestetskii 1953 (1) reinterpret Maxwell’s Equations as a one-photon Schrödinger Equation, and (2) quantify this photon with boson statistics.

2.9 Paleo-bosons

Baugh 2004 flexed the Heisenberg Lie algebra into the unitary Lie algebra. His homotopy can also be applied to the bosonic commutation relations by a change of basis.

Vilela space and Baugh space are related by an extension of the Weyl unitary trick. For brevity I write $V \oplus \mathbf{2}$ as V .

Definition 5 *A unitarization (relative to an orthonormal basis \mathcal{B} in a real quadratic space V with bilinear metric form $h_{\mu\nu}$) is a process that*

1. replaces

$$(V, h_{\mu\nu}) \rightarrow (V \otimes \mathbb{C}, h_{\mu^*\nu}), \quad (5)$$

a complex \dagger space with Hermitian sesquilinear metric form.

2. imbeds $V \subset V \otimes \mathbb{C}$ as the vectors with real coordinates in \mathcal{B} .

3. maps $\text{SO}(V) \rightarrow \text{SU}(V \otimes \mathbb{C})$, keeping the same matrix elements in the basis \mathcal{B} .
4. replaces Clifford generators γ_α obeying

$$\{\gamma_\mu, \gamma_\nu\} = 2h_{\mu\nu} \quad (6)$$

by Fermi generators $\gamma_\alpha, \gamma_\alpha^\dagger$ obeying

$$\begin{aligned} \{\gamma_\alpha^\dagger, \gamma_\beta\} &= 2h_{\alpha*\beta}, \\ \{\gamma_\alpha, \gamma_\beta\} &= 0, \\ \{\gamma_\alpha^\dagger, \gamma_\beta^\dagger\} &= 0. \end{aligned} \quad (7)$$

The grade-2 elements $\gamma_{\alpha\beta}$ are paleo-bosonic, of even exchange parity, but the underlying grade-1 generators γ_α are now odd, fermionic.

2.10 Fermionic aggregates

If the Fermi-algebraic quantum theory of gravity is correct, space-time is a singular limit of a fermionic aggregate. In any case, the main sources of gravity are fermionic, not bosonic, so we require the familiar algebra of an aggregate ME of fermionic entities E (soon to be events).

Assertion 2 *A spin 1/2 is a fermionic aggregate.*

Argument:

$$\text{M } V = \bigvee V = \text{Min Fermi } V = \text{Min Cliff Dup } V = \text{Spin Dup } V \quad (8)$$

with Dup of Definition 15, §9, and \dagger induced by that of V . To see the last equality in (8), recall that for Cartan 1935 and Chevalley 1954, $\bigvee V$ is the prototype spinor space for a neutral quadratic space isomorphic to $V \oplus V^D$. That is, the kets supporting the algebra Fermi V are the columns of a faithful irreducible representation of Cliff Dup V , hence spinors; and are also multivectors of the Grassmann algebra $\bigvee V$. \square

Usually one expresses gravity using spinors of the Minkowskian tangent space. This clashes with canonical quantum theory. Spinors require a bilinear form \ddagger , canonical quantum theory a Hermitian form \dagger . This discord already exists between Minkowski space-time and Hilbert space. Since we now regard the usual real Minkowski space-time as a singular limit of a quantum space-time with a Hermitian form, the discord disappears. We can express flexed gravity using multivectors over a fermion ket space.

$\sqrt{V} = \text{Min Fermi } V$ is a kind of square root of Fermi V . The dimension of Dup V is $2n$, the dimension of Cliff Dup V is 2^{2n} , and

$$\text{Dim Min Cliff Dup } V = \sqrt{2^{2n}} = 2^n. \quad (9)$$

Before encountering the Segal homotopy strategy, I vacillated between using Cliff V as algebra and as ket space. Now it is clear that the regular case Fermi $V = \text{Cliff Dup } V$ is the full matrix algebra, provided with intrinsic inner product, while its singular limit Grass V is only a ket space, requiring an external inner product.

2.11 Bosonization

It is especially easy to construct paleo-bosonic excitation quanta out of a fermionic aggregate.

Assertion 3 *A fermionic aggregate includes a paleo-bosonic aggregate.*

Argument The vectors γ_α of the Fermi algebra are fermionic generators, and the tensors $\gamma_{\alpha\beta}$ are unitary-group generators and therefore paleo-bosonic generators that have bosonic generators as singular limit. \square

The even sub-algebra $C_+ \subset C$ of Fermi statistics can be interpreted as the ket space of an aggregate composed of fermion-pairs γ_{ab} , of even exchange parity.

History H will be presented as a fermionic aggregate of events $E = \mu H$ that are in turn fermionic aggregates of mini-events $\mu E = \mu^2 H$.

2.12 The cosmological number problem

There is no sign of basic space-time aggregates intermediate between nuclear sizes and cosmic. Therefore we must leap from nuclear to cosmic sizes by a single fermionic quantification; just as classically one leaps from differentials to stellar distances by one integration.

Assertion 4 *The number of distinguishable events $E = \mu H$ in the maximal system history H is not greater than the dimensionality of the event ket space:*

$$N \leq \text{Dim } V E. \quad (10)$$

Argument Since $H = ME$, a basis of the ket space $V H$ can be composed of products of orthonormal kets in a basis of $V E$, each to some power. The number of kets with non-zero powers is an eigenvalue of N . This no greater than the number of kets in the basis. \square

Every model of event space since Archimedes is an instance of this assertion. Nowadays N has to be large enough to account not only for the vastness of space-time but also all for the fields that are assigned to the event in the singular limit of classical space-time. The successes of the continuum limit suggest that

$$P^5 1 = 2^{16} \sim 10^5 \ll N < \text{Dim } V E < P^6 1 = 2^{2^{16}} \sim 10^{10^{10}}. \quad (11)$$

N is so large that I assume that the event $E = \mu E$ too is composite. Its elements are mini-events $X = \mu E$, elsewhere called chronons. By Assumption 1 of basic Fermi statistics,

$$\text{Dim } V X \sim \log_2 \text{Dim } V E \sim 2^{16}. \quad (12)$$

We can reduce the event dimensionality if we restrict quantum theory to a logarithmically small part of the universe, leaving the rest for the meta-system including the experimenter. For example even if the number of events in the history of the universe is $NU \sim 10^{20\,000} \approx 2^{60\,000}$, the number NH of events in the maximum feasible quantum system history H is no more than about 60 000, very roughly speaking, and three levels of fermionic analysis bring us to binary elements: $E \sim P^3 2 = 65\,536$.

$$\text{Dim } V E \sim P^3 \mathbf{2}. \quad (13)$$

Nevertheless, for an aggregate of 60 000 fermionic events to be possible, the generic event must have kets of 60 000 components, and the chronon must have kets of 16 dimensions/ Macroscopic space-time extension singles out 4 of the 16, as though a condensation of many elements $\mu^2 E = M \mathbf{2}$.

2.13 General spin-statistics correlation

For any entity E let $X(E)$ be the *exchange parity* of E , the operator of an aggregate ME that exchanges two elements E by a homotopy. Let $W(E)$ be the (Wigner) spin parity of E , representing a continuous spatial rotation of one element E in the aggregate ME through 2π . $W(E)$ has eigenvalue $+1$ for integer spin and -1 for half-odd spin.

The observed spin-statistics correlation is

$$W(E) = X(E) \quad (14)$$

for all quanta E .

The quantifier M converts a single E into an entity composed of a variable number of E 's described by a spinor or multivector. The spin-statistics correlation is then the M - M correlation:

Assertion 5 (M-M correlation) *If $A = MB$ is a fermionic aggregate of quanta in space-time then its generic element is also a fermionic aggregate:*

$$A = MB \quad \text{implies} \quad B = MC \quad (15)$$

in nature.

Argument Since A is a fermionic aggregate, B must be a fermion. Then by the spin-statistics correlation, B is represented by a spinor. A spinor is a multivector. \square

The assertion simply restates the spin-statistics correlation.

M and Cliff both exponentiate the dimension:

$$\text{Dim } MC = \text{Dim } \text{Cliff } C = 2^{\text{Dim } C} \quad (16)$$

But their difference is essential. To iterate Cliff would seriously violate the spin-statistics correlation, To iterate M = Min Cliff Dup supports it:

Assertion 6 Basic Fermi statistics (Assumption 1) implies the spin-statistics correlation.

Argument According to Assumption 1 the quantifier M is applied at every level, and an M-M correlation exists throughout the vertical structure. The spin-statistics correlation is merely the special case where the middle level is that of quanta in space-time. \square

Conversely, if we accept the existence of one basic statistics, the M-M correlation implies that it is the Fermi statistics.

3 General quantum covariance

General relativity replaces Minkowski space as a model for event space by a space with much larger group:

Definition 6 *The **Einstein group** $E(\mathcal{M})$ is the group of diffeomorphisms $\mathcal{M} \rightarrow \mathcal{M}$ of the manifold \mathcal{M} whose points represent physical events, and whose metrical form is the gravitational or chronometric form $g_{\mu\nu}(x)$. The **Einstein Lie algebra** $dE(\mathcal{M})$ is the Lie algebra of the Einstein group.*

$dE(\mathcal{M})$ consists of the smooth real vector fields $X = (X^\mu(x)\partial_\mu)$ on \mathcal{M} , taken with the Lie product $X_1 \times X_2 := [X_1, X_2]$.

As Einstein based general relativity on covariance under the Einstein group, to maintain correspondence I base general quantum relativity on covariance under a quantized Einstein group \hat{E} , a flex of the Einstein group $E(\mathcal{M})$.

3.1 Non-locality

Ultimately $E(\mathcal{M})$ is compound and therefore singular because its elements respect the underlying points of the manifold, mapping points into points. This form of locality results in invariant subgroups and algebra ideals and must be eliminated in full quantization.

Flexing does this at an adjustable length. Vilela space and Baugh space (like Snyder space) already have a non-locality with range characterized by a new quantum constant of time. Since they are still without gravity, their theories might be called special quantum relativity, of the D and A series respectively. I carry this non-locality into general quantum relativity.

Quantizing the Einstein group is easier if we base it on an algebra instead of a manifold. One may redefine the Einstein group as the automorphism group of the algebra $A(\mathcal{M})$ of manifold coordinates:

$$E(\mathcal{M}) = \text{Auto } A(\mathcal{M}). \quad (17)$$

To quantize $E(\mathcal{M})$ I quantize the algebra $A(\mathcal{M})$ and take the automorphism group of the result. This drops smoothness and locality, which have no meaning for the quantized space and must re-emerge in the singular limit.

3.2 Renunciation of absolute space-time

Quantum events like those of Vilela space or Baugh space have, in each admissible frame, not only space-time coordinates but also momentum-energy and other coordinates, mixed by the invariance group. This is counterintuitive since it relativizes the construct of absolute space-time point that has pervaded physics since Galileo if not Aristotle.

Were Galileo and Einstein right on this point, there would exist entities underlying mechanics that have space-time coordinates but no momentum-energy coordinates: the space-time points. If we trace this notion back to the slate or the sandy beach on which Euclid and Archimedes developed and tested their ideas of geometry, we see that their “points” were actually bits of mineral. They had well-defined momenta that were small because they and the observer are both well-coupled to each other and to one large condensate, the Earth. Space-time is thus an extrapolated abstraction from the solid-seeming Earth. Therefore its elements too may have momentum coordinates that have been similarly suppressed. Quantum events are merely reclaiming the energy-momentum variables that were abstracted from them millennia ago. These energy-momentum variables may have zero vacuum expectation values, but they will inevitably contribute to the vacuum stress tensor, and possibly to dark matter and the cosmological constant.

3.3 Vilela space

I use the following familiar structures:

Definition 7 *The $2N + 1$ -dimensional Heisenberg Lie algebra $dH(N)$ is defined by the relations*

$$[p_\nu, q^\mu] = ir, \quad [r, p_\mu] = 0 = [q^\mu, r] \quad (18)$$

among $2N + 1$ Hermitian generators q^ν, p_ν, r .

The **Heisenberg group** $H(N)$ is the $2N + 1$ dimensional Lie group infinitesimally generated by $dH(N)$.

The **Heisenberg algebra** $A_H(N)$ is the algebra of bounded operators providing an irreducible faithful representation $R_{\hbar}H(N)$ with invariant $r = i\hbar$.

A **Heisenberg space** S_H is a quantum space whose coordinate algebra is the Heisenberg algebra and whose group of allowed coordinate transformations is the Heisenberg group.

Vilela 1994 flexed the Heisenberg Lie algebra to an orthogonal Lie algebra $dSO(6; s)$ with generators $o_{\alpha\beta}$. The Vilela coordinates are

$$\begin{aligned} \hat{x}^\alpha &= X o^{\alpha X}, \\ \hat{p}_\alpha &= P o_{\alpha Y}, \\ \hat{L}_{\alpha\beta} &= \hat{x}_\alpha \hat{p}_\beta - \hat{x}_\beta \hat{p}_\alpha = X P o_{\alpha\beta}, \\ \hat{i} &= l^{-1} o_{XY} \end{aligned} \quad (19)$$

with scale factors X and P having the units of length and momentum.

The quantum number l is the maximum eigenvalue of $-io_{XY}$ in the representation R_J .

The contraction to classical space-time includes the limits

$$X, P \rightarrow 0, \quad N, l \rightarrow \infty \quad (20)$$

and the freezing

$$-io_{XY} \approx l. \quad (21)$$

3.4 Unitarization

$dSO(6; s)$ can be imbedded isomorphically in a (pseudo-) unitary Lie algebra $dSU(6, s)$ by enlarging the underlying real vector space V with bilinear form $g_{\alpha\beta}$ to a complex vector space $V \otimes \mathbb{C}$ sesquilinear form $g_{\alpha^* \beta}$ with the same numerical coefficients in some frame \mathcal{B} :

$$g_{\alpha\beta} \stackrel{\mathcal{B}}{=} g_{\alpha^* \beta} \quad (22)$$

This process resembles the Weyl unitary trick; Let us call it *unitarization*. The generators of the orthogonal group $\text{SO}(V)$ are antisymmetric matrices $o_{\alpha\beta}$. Unitarization replaces these real antisymmetric matrices by matrices that have the same matrix elements in the frame \mathcal{B} , so I will continue to designate them by $o_{\alpha\beta}$. It appends an equal number of imaginary symmetric matrices $s_{\alpha\beta}$, and a smaller number of imaginary traceless diagonal matrices d_α , generators $\text{SU}(V \otimes \mathbb{C})$. It is convenient to use an overcomplete set of generators subject to $\sum_\alpha d_\alpha = 0$.

Instead of the usual infinite-dimensional representation R_\hbar of the Heisenberg Lie algebra $d\text{H}(\mathcal{N})$ by differential operators, one must choose among an infinite number of finite-dimensional representations $R_J d\text{SO}(\nu; \sigma)$ of the unitary Lie algebra $d\text{SU}(6; s)$ in $d\text{SU}(N; S)$, labeled by the appropriate collection J of quantum numbers. J determines the dimension N and signature S of the representation space. In what follows, a circumflexed variable is the R_J representative of the un-circumflexed variable.

3.5 Baugh space

Baugh space is then a unitarization of Vilela space. There is no need to rewrite the defining relations (19) but unitarization gives the symbols different meanings for Baugh space.

To imbed the Baugh algebra $d\text{SU}(V)$, and so the Vilela algebra, within a Fermi algebra $\text{Fermi } V$ (§9) one chooses an arbitrary orthonormal basis \mathcal{B} of vectors $|n\rangle \in V$ and writes $\langle \iota | n \rangle \in \text{Fermi } V$ for the corresponding creators. $\langle \iota | : V \rightarrow \text{Fermi } V$ is thus an operator-valued form characterizing fermionic statistics. $\langle \iota | n \rangle$ can be read as “create n ”. The corresponding annihilator can be written as $\langle n | \iota \rangle$, “ n -annihilate” with $|\iota\rangle : V^\text{D} \rightarrow \text{Fermi } V$ an operator-valued vector, acting from the right. Then the Fermi representative of any $u \in \text{GL}(V)$ is the well-known operator-valued expectation value

$$R_N u = \sum_{m,n} \langle \iota | n \rangle \langle n | u | m \rangle \langle m | \iota \rangle =: \langle \iota | u | \iota \rangle \in \text{Fermi } V. \quad (23)$$

We use this to imbed $\text{SU}(V)$ and $d\text{SU}(V)$ in $\text{Fermi } V$.

The Bose Lie algebra $\text{B}(V)$ is generated by vectors $v \in V$ and $v^\dagger \in V^\text{D}$ with $[u, v^\dagger] = v^\dagger \bullet u$ and $[u, v] = 0$. Obviously $\text{B}(V) \leftrightarrow \text{H}(V)$: the Bose and Heisenberg Lie algebras are isomorphic. I apply our prior flex of $\text{H}(V)$ to $\text{B}(V)$:

To flex $\text{B}(V) \leftarrow \text{SU}(V \oplus \mathbf{2}) \subset \text{Fermi}(V \oplus \mathbf{2})$, append two additional basis elements γ_X, γ_Y orthogonal to V to form the vector space $V'' := V \oplus \mathbf{2}$. Then flex

$$v \rightarrow \widehat{v} := \langle \iota | v \rangle \langle X | \iota \rangle \in \text{Fermi}(V''),$$

$$v^\dagger \rightarrow \widehat{v}_k^\dagger := \langle \iota|Y\rangle\langle v|\iota\rangle \in \text{Fermi}(V''). \quad (24)$$

The flexed commutation relations for vectors $v, w \in V$ follow:

$$\begin{aligned} [\widehat{v}, \widehat{w}] &= 0, \\ [\widehat{v}^\dagger, \widehat{w}] &= \langle \iota|Y\rangle\langle X|\iota\rangle\langle v|w\rangle, \\ [\widehat{v}^\dagger, \widehat{w}^\dagger] &= 0, \end{aligned} \quad (25)$$

Other variables vanish or freeze in the singular limit of the usual bosonic statistics. This limit first “freezes” the imaginary generator io_{XY} to a small sector of its ket space where io_{XY} is close to its extreme eigenvalue N . The sector must be large enough to support all the kets of the singular limit, yet small enough so that $io_{XY} \approx N$. Typically I consider neighborhoods containing $O(\sqrt{N})$ eigenvalues of io_{XY} or fewer as $N \rightarrow \infty$ in the canonical limit.

3.6 General quantum relativity group

For purposes of quantization, let us redefine the general relativity group (diffeomorphism group) in more algebraic language:

Definition 8 *The general relativity group or Einstein group $E(\mathcal{N})$ of a classical space-time \mathcal{N} (redefined) is the group of smooth local automorphisms of the commutative coordinate algebra $\mathbf{A}(\mathcal{N})$ of all coordinate functions on \mathcal{N} .*

Then to quantize the Einstein group it remains only to quantize the algebra it acts on.

General relativity (or general covariance) contrasts with special relativity in that it implicitly assumes that the coordinates on physical space make up a commutative algebra. The fully quantum correspondent is a full matrix algebra. The Assumption 1 of basic Fermi statistics implies that this is a Fermi algebra, that of the ket space of the mini-event $M E =: X: \mathbf{A} E = \text{Fermi } V X$.

Assumption 2 (General quantum relativity) *The event is a Fermi aggregate. The generalized event coordinate is an operator on the ket space of the aggregate. The general quantum relativity group is the automorphism group of the algebra of event coordinates.*

I write $E = M X$ for the event and G for the group. G must be distinguished from the automorphism group of the graded Lie algebra $d\text{SU}(V E) \subset G$, a much smaller group appropriate to special quantum relativity.

It follows that the ket space of the event is $V E = \bigvee V X$, with dimensionality $\nu := \text{Dim } V E = 2^{\text{Dim } V X}$. The special quantum coordinates of the

event form not an algebra but merely the Lie algebra $d\text{SU}(VE) \subset \text{Fermi } VX$, imbedded in the Fermi algebra by the representation

$$d\text{SU}(VE) \ni u \mapsto \langle \iota | u | \iota \rangle \in \text{Fermi } VE \quad (26)$$

The general quantum coordinates form the algebra $\text{Fermi } VX$. The general quantum relativity group G is the group of the regular elements of $\text{Fermi } VX$, modulo its center \mathbb{C} .

This brings the groups of general relativity and quantum theory into close alignment. Both are plausible contractions of one general quantum relativity group $\text{SU}(\nu; \sigma)$ with event ket space $VE = \mathbb{C}(\nu; \sigma)$, a complex \dagger space of dimension N and signature S .

The main sequence of contractions or singular limits now proposed is

General quantum space \rightarrow Baugh space \rightarrow Heisenberg space \rightarrow
Minkowski space.

Classical and semiclassical general relativity lie on another line of contractions that dangles from the left-most space of this sequence. Spinors arise as singular limits of multivectors.

This puts the diffeomorphism groups of general relativity and the unitary groups of quantum theory into precise alignment. Both are plausible contractions of the general quantum relativity group, along different contraction paths.

By the assumption of Fermi statistics the kets of the physical event and the physical history are multivectors and therefore spinors. There is no free choice of representation, as there is for Vilenka space, Baugh space, and paleo-bosonic statistics. Fermi algebras have unique faithful irreducible representation.

For the fermionic event $E = MX$ with ket space $V = VE$, the quantum history $H = ME$ has ket space $VH = VME = MV$ and coordinate algebra $AH = \text{Fermi } V$.

Every Fermi algebra $\text{Fermi } V$ has a Hermitian norm

$$\|x\| := \frac{1}{N} \text{Tr } x^\dagger x \quad (27)$$

and a quadratic form

$$Q(x) := \frac{1}{N} \text{Tr } x^2 \quad (28)$$

This quadratic form is indefinite, as is needed for physics, of signature $N(N+1)/2 - N(N-1)/2 = N$, the square root of its dimension. For example, it is a Minkowskian form of signature 2 on the 2×2 matrices

4 General quantum kinematics

Einstein used a scalar-valued quadratic form $v^{\mu'}(x)g_{\mu'\mu}(x)v^\mu(x)$ on vectors to describe gravity. In canonical quantum gravity this form would be operator-valued.

Assumption 3 (Theory of gravity) *General quantum gravity is described by the operator-valued quadratic form v^2 , where v is a cliffor (Clifford element) that has $v(x)$ as singular limit and the product is the Clifford product.*

Argument A plausibility argument occupies the rest of §4.

The gravitational quadratic form maps vectors to scalars. But vector fields in turn are derivations (differentiators) on scalar fields. Therefore in order to quantize gravity I first quantize the scalar fields, then the vector fields, and finally gravity.

4.1 Clifford ring of classical gravity

For completeness I review the Clifford ring $C(\mathcal{M})$ of classical gravity on a space-time manifold \mathcal{M} , giving its scalars, vectors, and product, before quantizing it.

By a local ring on a manifold I mean one with local (associative) product and sum; that is, the values of the sum and product at a point are determined by the values of their arguments at that point.

Definition 9 *The gravitational Clifford ring $C(\mathcal{M})$ of a classical gravitational manifold is a local Clifford ring with these properties:*

1. *The scalars of $C_0(\mathcal{M}) \subset C(\mathcal{M})$ are the smooth scalar functions $\mathcal{M} \rightarrow \mathbb{R}$.*
2. *The vectors of $C_1(\mathcal{M}) \subset C(\mathcal{M})$ are the vector fields, the derivations on $C_0(\mathcal{M})$.*
3. *(Clifford law) The square of a vector is a scalar. This scalar then defines a local quadratic form in the vector,*

$$\|v(x)\| = g_{\mu\nu}(x)v^\mu(x)v^\nu(x). \quad (29)$$

4. *This is the gravitational quadratic form of proper time.*

In other words, classical gravity is the part of the structure tensor of the associative Clifford ring of space-time that couples two classical vector fields to a classical scalar field; and it determines the rest of the structure tensor.

One finds the gravitational quadratic form at a point by limiting the vector fields to a small neighborhood of the point and measuring proper times. This requires arbitrarily small clocks and rods, so it does not work below a certain size.

\mathcal{M} must have suitable global topology (second Stiefel-Whitney class 0) for its gravitational Clifford ring $C(\mathcal{M})$ to admit a globally defined spinor module $\Psi(\mathcal{M})$ with $C(\mathcal{M}) = \text{Endo } \Psi(\mathcal{M}) = \Psi \otimes \Psi^\dagger$.

4.2 Quantized scalars

Recall that the algebra $A(\mathcal{N}) = C_0(\mathcal{N})$ of scalars on a space-time manifold \mathcal{N} is the grade-0 part of the Clifford ring $C(\mathcal{N}) := \text{Cliff}(\mathcal{N})$ of §4.1. It has the following properties:

1. $A(\mathcal{N})$ is an associative unital commutative real algebra.
2. $A(\mathcal{N})$ contains and is generated by the coordinate functions x^μ of one frame and the imaginary unit i .
3. $A(\mathcal{N})$ is invariant under Einstein $E(\mathcal{N})$.
4. $A(\mathcal{N})$ is commutative.

The third condition incorporates the principle of equivalence. The fourth condition, commutativity, implies that $A(\mathcal{N})$ is the algebra of functions on the state space of some classical object, here the generic event of \mathcal{N} . The quantum correspondents would seem to be:

Assumption 4 *The ring \hat{A} of scalar functions in general quantum relativity regarded as quantum coordinates of the quantum event, has the following properties relative to the Clifford and Fermi algebra $\hat{C} = \text{Cliff Dup } V = \text{Fermi } V \supset \hat{A}$:*

1. \hat{A} is a full matrix algebra.
2. \hat{A} includes the quantum coordinate functions \hat{x}^μ and imaginary \hat{i} .
3. \hat{A} is invariant under G .
4. \hat{A} is minimal in the above respects.

By conditions 2 and 3, $\hat{A} \supset \hat{C}_+$, the even-grade subalgebra of \hat{C} , do thst necessarily

$$\frac{1}{2} \text{Dim } \hat{C}_+ \leq \text{Dim } \hat{A} \leq \text{Dim } \hat{C}. \quad (30)$$

I satisfy these condition by assuming $\hat{A} = \hat{C}$: The quantum scalars form the entire Clifford-Fermi algebra. This obviously satisfies all the conditions but minimality, which I conjecture to hold.

The coordinate contraction $\hat{x}^\mu \rightarrow \tilde{x}^\mu, \hat{i} \rightarrow i\hbar$ induces an algebra contraction $\hat{A} \rightarrow A(\mathcal{N})$.

The grade-2 cliffors of the form $\gamma_{\mu X}, \gamma_{XY} \in \hat{C}$ represent the action of infinitesimal orthogonal unitary transformations $o_{\mu X}, o_{XY} : V \rightarrow V$ upon multivectors of $\bigvee V$. The special quantum coordinate \hat{x}^μ is

$$\hat{x}^\mu = X\gamma^{\mu X} : \hat{C} \rightarrow \hat{C}. \quad (31)$$

The coordinates commute in the contraction limit $X \rightarrow 0, N \rightarrow \infty$. One may perform the contraction to the canonical limit by changing two generators of the Clifford-Fermi algebra \hat{C} from γ_X, γ_Y to $X\gamma_X, P\gamma_Y$, freezing γ_{XY}^2 , and taking the limit 20 of the structure tensor. The freezing consists in restricting the ket space to a subspace in which $(\gamma_{XY})^2$ is relatively near its extreme value $-(N)^2$. Then the general quantum operator

$$\hat{i}\hbar := \frac{\gamma_{XY}}{N}. \quad (32)$$

reduces to the usual quantum imaginary $i\hbar$.

This gives \hat{A} a large new center in the limit of classical space-time, the ring of complex scalars $\text{Cliff}_0(\mathcal{N}) \subset \hat{C}$.

4.3 Quantized vectors

The defining property of the contracted vector fields $\gamma_\mu(x) \in \text{Cliff}_1(\mathcal{N})$ is that they are the derivations of the ring of scalars $\text{Cliff}_0(\mathcal{N})$ over the field \mathbb{R} :

$$\text{Cliff}_1(\mathcal{N}) := D \text{Cliff}_0(\mathcal{N}). \quad (33)$$

They form a Lie ring over the algebra of scalars $\text{Cliff}_0(\mathcal{N})$. I therefore define:

Definition 10 *The general quantum vectors are elements of the Lie algebra of derivations $\Delta\hat{A}$ of the general quantum scalar algebra $\hat{A} = \hat{C}$.*

That is, the same Clifford-Fermi algebra \hat{C} that describes quantum scalars by its associative structure describes quantum vectors by its commutator structure. This vector construction is invariant under the quantized Einstein group and contracts to the usual concept in the space-time limit.

4.4 General quantized gravity

By the MM hypothesis (§2.5), the ket spaces of the quantum history H , the quantum event $E = \mu H$, and the quantum mini-event $X = \mu^2 H$ are multivector and spinor spaces:

$$\bigvee H = \bigvee \bigvee \mu H = \bigvee^2 \bigvee \mu^2 H. \quad (34)$$

In the general quantum theory I take the gravity form to be the quadratic form of the Clifford product of the event algebra $\widehat{C} = \text{Fermi } \mathbb{V}E = \text{Cliff Dup } V$. That is, the general quantum operator-valued bilinear form $\widehat{g}_{\circ\circ}$ of gravity is the tensor representing the bilinear form $\widehat{g}(u) = u^2 = u^2$, the square of the cliffors that contract to vector fields in the limit of classical space-time. Let A, B, \dots be collective indices of the event space. Then

$$\widehat{g}_{\{A|B\}} := \text{frac}12\{\gamma_A, \gamma_B\}. \quad (35)$$

The skew-symmetric part of the same product defines the infinitesimal generator $\gamma_{[A|B]} \in d\widehat{\mathcal{E}} = d\text{Iso}(\widehat{C})$ of the general quantum Einstein group:

$$\gamma_{[A|B]} = \frac{1}{2}[\gamma_A, \gamma_B] = -\gamma_{[B|A]} \quad (36)$$

In this general quantum kinematics the gravitational field is not a function on a space-time manifold. The c space-time manifold is an organization or condensation of the quantum event space.

5 General quantum dynamics

I may still assume that the dynamical history multivector is an exponential

$$\Omega = e^{iA/\hbar} \in \bigvee \mathbb{V} E \quad (37)$$

where now A is an action multivector.

Here I make another tactical decision. Rather than quantizethe Hilbert action, which is not quantum general covariant, I seek quantum general covariant second-order dynamical equations.

What corresponds to general covariance is

Assumption 5 (General quantum covariance) *The action A is invariant under the group of transformations $\mathbb{V}H \rightarrow \mathbb{V}H$ induced by the general quantum covariance group $G : \mathbb{V}E \rightarrow \mathbb{V}E$ through the Fermi quantification relation $H = ME$.*

Assertion 7 *The action A is a polynomial in the Casimir operators of the unitary group G .*

Argument Clear. \square

Einstein assumed that the dynamical law was. expressed by second-order wave equations. This led to an action that is second order in p . but not in x . If we imitate him too closely we would break the o_{XY} symmetry between x and p , violating general quantum covariance.

Assumption 6 *The length and momentum scales of experiments so far, X_{Exp} and P_{Exp} , are related to the quantum units by*

$$X \ll X_{\text{Exp}}, \quad P \gg P_{\text{Exp}} \quad (38)$$

This would explain why we have been led to a singular theory with x -locality and p -non-locality. Now we must give up the x -locality, restoring $x \leftrightarrow p$ symmetry. Then I can suppose, following Einstein and Hilbert, that

Assumption 7 *The action is second-order in the generators of the group $\text{SU}(VE)$.*

This singles out the Casimir invariant of the unitary group of the quantum event E as represented in the unitary group of the quantum history H . If u_A is a basis for $d\text{SU}(E)$ and $\hat{u}_A = \langle \iota | u_A | \iota \rangle \in \mathbf{A}H$ is the representation of u_A in $\mathbf{A}H$ then the Killing form is

$$K_{AB} = \text{Tr } \Delta \hat{u}_A \Delta \hat{u}_B, \quad (39)$$

and the action is the Casimir invariant with a multiplier coupling constant m :

$$A = m K^{AB} \hat{u}_A \hat{u}_B. \quad (40)$$

6 Discussion

6.1 Roads to the quantum event

The photon, the quantum of the electromagnetic oscillator, had to be verified several times to be generally accepted. The quantum event and sub-event will require at least as much scrutiny. On the one hand, they attack even deeper continuity assumptions than the photon did; on the other hand, physicists are have more experience with such reformation processes today. The three main roads to the photon were:

1. Regularization. Planck first introduced the quantum constant h to eliminate the infinite classical heat capacity of electromagnetic cavities.
2. One-photon observations. Einstein recognized h as the action quantum of a single photon and estimated it independently from the photoelectric effect. Compton confirmed the photon and estimated h yet again by bouncing photons one at a time off free electrons.
3. Quantization. Dirac deduced the photon from the canonical commutation relations for the electromagnetic field.

It is harder to see one event than one photon. Space-time is stiff while cavity photons form an ideal gas, so events in a cavity are strongly coupled while the photons in the same cavity are weakly coupled. To predict quantum effects that are presently observable we must formulate a theory of many strongly-coupled events before we ever isolate a single one. We did not have this problem with the photon.

Since we cannot reach the quantum event by road 2 yet, I have approached it by roads 1 and 3.

6.2 The kinematics of general quantum gravity

I have argued that the macroscopic Clifford ring $C(\mathcal{M})$ of classical gravity on a space-time manifold \mathcal{M} is a singular limit of an underlying Clifford algebra \hat{C} of general quantum gravity; that the macroscopic Lie product of classical gravity is a singular limit of the Lie algebra of the even-grade cliffors of \hat{C} , with the commutator as Lie product; that the classical gravitational field is a singular limit of the general quantum operator-valued quadratic form defined by the Clifford product of \hat{C} .

The Einstein group respects the Lie product but not the ring product. The quantized Einstein group is therefore the special unitary group $SU(E)$ of the event ket space. The correspondence from $SU(E)$ to the Einstein group is the main tool of this paper. It leads to the general quantum gravitational kinematics of (35) and the dynamics of (40).

General relativity renounced the idea that space-time coordinates have immediate metrical meaning but retained the absolute points themselves. General quantum relativity renounces absolute space-time points for a unified quantum concept of space-time, angular momentum, and momentum-energy. The methods apply to gauge theories in general.

Dynamics is defined here not by a one-parameter group of unitary operators generated by the Hamiltonian but by a ket defining a probability amplitude for quantum histories, corresponding to the classical action.

6.3 Nonlocality

To take this extension of relativity seriously we must overcome the enormous apparent difference between space-time coordinates and momenta in our current experience. The difference boils down to the usual assumption that Nature can make jumps in p but not in x ; or that fields are diagonal in x but not in p .

In general quantum relativity I must suppose that this is a broken $x \leftrightarrow p$ symmetry. When we restore the broken $x \leftrightarrow p$ symmetry we lose locality in x as well as p . Infinitesimal locality is not even defined for general quantum

variables, which have discrete spectra. Quantum theory already permits us to interchange x and p by Fourier transformation. In Vilela space, x and p are interchanged by the operator o_{XY} ; in Baugh space by o_{XY} and s_{XY} . Since breaking o_{XY} invariance might break i invariance, it is good that we have s_{XY} invariance to break instead: another amenity of the A series.

The spectral gap X in x is also a measure of the non-local jumps in x . The difference in locality that we currently see reflects a real difference in the ranges X and P of the quanta of x and p on the scale of present quantum experiments.

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8 Appendix: Clifford algebra

I use a Clifford construct that covers the following cases:

1. The Clifford algebra of Chevalley, with an arbitrary field of coefficients and an arbitrary quadratic space.
2. The Clifford algebra of the usual quantum theory of spin, with complex field and Minkowskian quadratic space.
3. The Clifford ring of general relativity, with coefficients in the ring of scalar fields, which is not a field, and quadratic space composed of the vector fields.
4. The Clifford algebra of every Fermi algebra, with complex field and complex neutral quadratic space.
5. The Clifford algebra of classical set theory, with scalars in the binary field 2, and vectors in the second power set of an arbitrary finite universe of discourse. Its Clifford product has $\gamma^2 = 1 = \emptyset$ and is a XOR operation; its Clifford sum has $\gamma + \gamma = 0 = \text{the undefined}$ and is the addition of Boole as extended by Pierce.

Let C_1 be a \dagger module called the vectors. Call its commutative unital ring of coefficients C_0 the scalars.

Definition 11 A **Clifford ring** over C_1 is a ring $C \supset C_1$ generated by C_1 that obeys Clifford's Law in the form

The square of any vector is a scalar .

The Clifford form is the quadratic form $N : C_1 \rightarrow C_0$, $v \mapsto v^2 =: \|v\|$ on C_1 , **Cliffors** are elements of C . A **Clifford algebra** is a Clifford ring whose scalars form a field.

C defines a bilinear inner product for any vectors $u, v \in C_1$:

$$u \cdot v = \frac{1}{2}\{u, v\}. \quad (41)$$

I assume here that this bilinear form is regular.

A Clifford ring with product $u \sqcup v$ defines a Grassmann ring with the same elements with Grassmann product $u \vee v$ such that for vectors $u, v \in C_1$

$$u \sqcup v = u \cdot v + u \vee v. \quad (42)$$

The Grassmann ring has a well-known \mathbf{N} -valued grade. This grades the Clifford ring too. Insofar as its Grassmann algebra can be interpreted as the algebra of a fermionic aggregate, so can a Clifford algebra.

The Clifford norm is invariant under automorphisms of the Clifford algebra. Its signature is determined by the signature and dimension of the quadratic space C_1 .

Definition 12 The **spinor space** $\text{Spin } V$ over a quadratic space V consists of the columns of (a faithful irreducible matrix representation of) the Clifford algebra $\text{Cliff } V$ [Budinich and Trautman 1988]:

$$\text{Spin } V = \text{Min Cliff } V \quad (43)$$

9 Appendix:Fermi algebra

The Fermi algebra of Fermi-Dirac statistics is a Clifford algebra on the A series, close in spirit to Cartan's original construction of spinors for the D .

Definition 13 Let K be a ring with a quadratic norm $\|k\| = k^\dagger k$. Let V be a module over K (usually a vector space over a field). Then V^\dagger designates the **dual module** of K -linear mappings $V \rightarrow K$. $\dagger : V \rightarrow V^\dagger$ designates a singled-out non-singular anti-linear involutory map $V \rightarrow V^\dagger$, $V^\dagger \rightarrow V$, $K \rightarrow K$, agreeing with the given \dagger on K . A \dagger **module** is a module endowed with a \dagger . A \dagger **algebra** is a \dagger space with an algebra product of which \dagger is an involutory anti-automorphism: $(ab)^\dagger = b^\dagger a^\dagger$. $\text{Dim } V$ is the dimension of V over K .

Let V be a \dagger space over \mathbb{C} , and let $DV = V^{\mathsf{D}}$ be the dual space of linear maps $V \rightarrow \mathbb{C}$. Let $f \bullet x := f(x)$ represent application of a map f to its argument x .

Definition 14 • Fermi V , the **Fermi algebra** over V , is the \dagger algebra generated by $V, V^{\mathsf{D}} \subset \text{Fermi } V$ subject to the Fermi-Dirac relations:

$$\forall v, w \in V \quad \left| \quad (v + w^{\dagger})^2 = \|v + w^{\dagger}\| = w^{\dagger} \bullet v + v^{\dagger} \bullet w. \right. \quad (44)$$

$$\bullet \forall f \in \text{Fermi } V \quad \left| \quad \|f\| := \text{Re } f^2. \right.$$

The space MV of **multivectors** is the \dagger space $\bigvee V \subset \text{Fermi } V$ with the $\dagger : MV \leftrightarrow MV^{\mathsf{D}}$ induced by the \dagger on V . A is the endomorphism algebra of V and MA is the endomorphism algebra of MV . **Multi-operators** are operators on multivectors; elements of MA . • For any unitary transformation $u : V \rightarrow V$, $Mu : MV \rightarrow MV$ is the induced automorphism and $Su \in V \vee V^{\mathsf{D}} \subset \text{Fermi } V$ is any element generating Mu as an inner automorphism: $Mu = Su v (Su)^{-1}$. \square

Standard interpretation:

- V is the ket space of a fermionic quantum entity E , representing input channels for the entity. A \dagger on V induces one on $\text{Fermi } V$.
- MV is the ket space of a many- E aggregate ME .
- $\text{Fermi } V$ is the coordinate algebra of the aggregate ME .
- $\|f\|$ is the transition probability amplitude for the process represented by $f^2 = f \circ f$. \square

$$\vdash: \quad \text{Dim } MV = 2^{\text{Dim } V}.$$

A Fermi algebra is a Clifford algebra:

Definition 15 The **duplication space** of V is

$$\text{Dup } V := V \oplus V^{\mathsf{D}} \quad (45)$$

with quadratic form \ddagger defined for any $v \in V, w \in V^{\mathsf{D}}, (v, w) \in \text{Dup } V$ by

$$\|(v, w)\| = (v, w)^{\ddagger}(v, w) := w(v). \quad (46)$$

The duplication space was introduced by Saller 2006a as the *quantum space* of the system. $\text{Dup } V$ has both a neutral quadratic form and a hermitian form induced by that of V .

Assertion 8 Every Fermi algebra is a Clifford algebra.

Argument

$$\text{Fermi } V = \text{Cliff Dup } V \quad \square \quad (47)$$

The \dagger on the multivector (spinor) space MV is required to be invariant under $SU(V)$. This singles out the \dagger on $\bigvee V$ induced by the \dagger on V , with the property that

$$\forall u, w \in MV \quad \bigg| \quad (u \vee w)^\dagger = w^\dagger \vee u^\dagger. \quad (48)$$

This is positive definite. The indefinite metrics that we need in physics cannot come from this \dagger . They may arise from the neutral quadratic form on $\text{Dup } V$.

The first natural Fermi multivector spaces $M^L\mathbb{C}$:

$$M^0\mathbb{C} = \mathbb{C}, \quad (49)$$

$$M^1\mathbb{C} = 2\mathbb{C}, \quad (50)$$

$$M^2\mathbb{C} = 4\mathbb{C}, \quad (51)$$

$$M^3\mathbb{C} = 16\mathbb{C}, \quad (52)$$

$$\vdots \quad (53)$$

$$M^L\mathbb{C} = P^L 1. \quad (54)$$

M^2 is a four-dimensional space with Minkowskian quadratic form.

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